

TBA and Y-system for planar $\text{AdS}_4/\text{CFT}_3$

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Abstract

We conjecture the set of asymptotic Bethe Ansatz equations for the *mirror* model of the $\text{AdS}_4 \times \mathbb{CP}^3$ string theory, corresponding to the planar $\mathcal{N} = 6$ superconformal Chern-Simons gauge theory in three dimensions. Hence, we derive the (vacuum energy) thermodynamic Bethe Ansatz equations and the Y-system describing the *direct* $\text{AdS}_4/\text{CFT}_3$ string theory.

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1 Introductory remarks

The $\text{AdS}_4/\text{CFT}_3$ strong/weak coupling duality relates conjecturally the type IIA superstring theory on curved space-time $\text{AdS}_4 \times \mathbb{CP}^3$ to its boundary theory, namely the conformal $\mathcal{N} = 6$ super Chern-Simons (SCS) gauge theory in three dimensions [1]. Similarly to the previously discovered $\text{AdS}_5/\text{CFT}_4$ correspondence [2], the initial hints of integrability both in perturbative calculations of anomalous dimensions (of single trace composite operators in the $SU(2) \times SU(2)$ sector) of SCS [3] and in energies of string configurations on the $\text{AdS}_4 \times \mathbb{CP}^3$ background [4] have clearly marked the beginning of a new and fascinating research topic. In fact, integrability structures and tools typically imply important enhancements of non-perturbative or exact calculations.

By mimicking the SYM_4 results and Dynkin diagram structure [5], a set of all-loop Bethe Ansatz (BA) equations valid in the asymptotic regime of large quantum numbers (e.g. size or R-charge L , Lorentz spin, etc.) has been proposed for all the SCS sectors [6]. In [7] a scattering matrix was devised in such a way that the corresponding Bethe-Yang equations match the BA equations of [6]. For the particular interest of the present work, the finite size corrections implied by this scattering matrix have been investigated and compared [8] with string theory results [9], going beyond the asymptotic regime and confirming the *bona fide* feature of the S-matrix. Yet, for *small* size L the Lüscher method does not allow a feasible extension to all loops and thus turns out to be unfit for a large class of interesting operators.

On the contrary, the Thermodynamic Bethe Ansatz (TBA) idea recently succeeded in providing a set of infinite integral equations, which should govern *in principle* the spectrum of anomalous dimensions non-perturbatively and for general values of the quantum numbers [10, 11, 12]. The statement ‘*in principle*’ results from the consideration that the TBA procedure involves a minimisation to the ground state energy, although the first few excited states may be extracted from it. In fact, the TBA method for calculating the free energy of statistical field theories at temperature T dates back to Yang and Yang [13], and provides for a minimisation procedure for both Fermi and Bose statistics [13]. Furthermore, the idea of combining the TBA with the modular transformation (or double Wick rotation) exchanging space and time in quantum integrable 2D relativistic massive field theories belongs to Al.B. Zamolodchikov [14].

The TBA procedure may be briefly described in the following way. Let us assume that the original (*direct*) theory is defined on a torus space-time geometry, the space is a ring of finite circumference L , while time runs over a very long circumference $R \rightarrow \infty$. Modular transformation amounts in exchanging space and time, thus defining a mirror theory living on a space segment of length $R \rightarrow \infty$. For this theory the asymptotic *mirror* Bethe-Yang equations are exact. Time is compactified on a circumference L that can be interpreted as the inverse of the temperature $T = 1/L$ and the Yang-Yang thermodynamic Bethe Ansatz procedure [13] can be used to find the minimum free energy of the mirror theory or equivalently the vacuum energy of the original *direct* theory confined on a ring of circumference L . In relativistic quantum field theories the ground state TBA equations have been generalised by [15, 16] to excited states.

Starting from the equations for the ground state, excited states were obtained in [16] through a process of analytic continuation in the particle masses and by considering the points where the singularities of the integrand cross the integration contour.

The possibility to extend the procedure of [15, 16] to the non-relativistic $\text{AdS}_5 \times \text{S}^5$ model was anticipated in the ‘Partial conclusions and remarks’ section in [10] by emphasising the appearance of new driving terms of the form $\sum_i \ln S(u_i, u)$ as residues of the convolution integrals, with S suitable scattering matrix elements¹. This extension was studied in more details for the excited states of the $sl(2)$ sector in SYM_4 in two different articles, [11] and [19]. In the latter reference some problems and modifications of the equations when varying the ’t Hooft coupling g have been highlighted, thus making the problem still more puzzling.

In the context of the $\text{AdS}_5/\text{CFT}_4$ correspondence, the mirror theory was brilliantly introduced and analysed in [20] on the basis of previous studies on the analytic properties of the direct S-matrix (like for instance [21, 22, 23, 24]). Recently, the string hypothesis, which is the basis for the TBA procedure, was proposed and studied within the large-size thermodynamic limit [25].

Here, bearing this in mind, we conjecture a set of Bethe Ansatz equations for the mirror theory of $\text{AdS}_4/\text{CFT}_3$. The latter show bound states of particles of type A with particles of type B in their $sl(2|1)$ grading. Therefore, we find profitable to formulate the string hypothesis in this grading, though it differs neatly from that proposed in [25]. Eventually, we implement the string hypothesis to derive the ground state thermodynamic Bethe Ansatz equations (for the TBA variables or *pseudoenergies*). From these equations we also deduce the Y-system, namely *universal* functional equations among the Y-functions, which are nothing but the exponential of the pseudoenergies. Although it is evident that we shall suffer an information loss through this step, the Y-system is believed to be universal in the sense of remaining the same for excited state equations as well.

The rest of this paper is organised as follows. Section 2 contains a brief summary on the magnon dispersion relations for the string $\text{AdS}_4/\text{CFT}_3$ and its mirror theory (also with the uniformisation variable). The proposed all-loop Bethe Ansatz equations for the mirror theory and the corresponding string hypothesis in the $sl(2|1)$ grading are discussed in sections 3, 4 and 5. The thermodynamic Bethe Ansatz procedure is implemented and the corresponding Y-system derived respectively in sections 6 and 7. Section 8 contains some general concluding remarks and a short discussion on the connection between our Y-system and that proposed, for the same theory, in [26]. Finally, in Appendix A we summarise the S-matrix elements appearing in the Bethe Ansatz equations and defining the TBA kernels, together with the functional identities enjoyed by the latter.

2 Dispersion relations

The dispersion relation of the $\text{AdS}_4 \times \mathbb{CP}^3$ string theory concerns two species, A and B , of ‘magnon’ excitations with energy and momentum (H^α, p^α) ($\alpha = A, B$) related by

$$H^\alpha = \frac{1}{2} \sqrt{1 + 16h^2(\lambda) \sin^2 \frac{p^\alpha}{2}} , \quad (2.1)$$

¹This mechanism becomes even simpler and straightforward in the (different) NLIE set-up when starting from the microscopic Bethe Ansatz description [17] (cf. also [18] about the SYM_4 non-relativistic (but asymptotic) case).

and total energy $H = \sum_{\alpha=A,B} H^\alpha$. Upon the analytic continuation $p^\alpha \rightarrow i\tilde{H}^\alpha$ and $H^\alpha \rightarrow i\tilde{p}^\alpha$ we obtain the mirror theory dispersion relations

$$\tilde{H}^\alpha = 2 \operatorname{arcsinh} \frac{\sqrt{1 + 4(\tilde{p}^\alpha)^2}}{4h(\lambda)} . \quad (2.2)$$

The dispersion relation (2.1) can be uniformised in terms of Jacobi elliptic functions:

$$p = 2\operatorname{am}(z, k), \quad \sin \frac{p}{2} = \operatorname{sn}(z, k), \quad H = \frac{\operatorname{dn}(z, k)}{2}, \quad (2.3)$$

with $k = -16h^2(\lambda)$, and then parameterised by an elliptic curve with periods

$$2\omega_1 = 4K(k), \quad 2\omega_2 = 4iK(1-k) - 4K(k) .$$

For a real particle with momentum p we know that

$$p = 2i \operatorname{arcsinh} \frac{\sqrt{1 + 4(\tilde{p}^\alpha)^2}}{4h(\lambda)} = 2\operatorname{am}(z, k) \quad (2.4)$$

and therefore

$$\tilde{p} = -\frac{i}{2} \operatorname{dn}(z, k), \quad (2.5)$$

that is real only if we shift the variable z by a quarter of the imaginary period. Thus, if we shift H by $\omega_2/2$, then $H \rightarrow i\tilde{p}$, while p becomes

$$p = 2\operatorname{am}z \rightarrow 2i \operatorname{arccoth} \frac{\sqrt{1-k}}{\operatorname{dn}z} = i\tilde{H} . \quad (2.6)$$

In fact,

$$2 \operatorname{arccoth} \frac{\sqrt{1-k}}{\operatorname{dn}z} = 2 \operatorname{arcsinh} \frac{\sqrt{1 + 4(\tilde{p}^\alpha)^2}}{4h(\lambda)} \quad (2.7)$$

gives the same expression of (2.2).

3 The scattering matrix

The $\operatorname{AdS}_4/\operatorname{CFT}_3$ S -matrix was proposed in [7]. As there are two sets of momentum carrying excitations, called A - and B -particles, each of which form a four-dimensional representation of $SU(2|2)$, the conjectured exact S -matrix has the following factorised structure:

$$\begin{aligned} S_{AA}(p_1, p_2) &= S_{BB}(p_1, p_2) = S_0(p_1, p_2) S(p_1, p_2) , \\ S_{AB}(p_1, p_2) &= S_{BA}(p_1, p_2) = \tilde{S}_0(p_1, p_2) S(p_1, p_2) , \end{aligned} \quad (3.1)$$

where S is the $SU(2|2)$ $\operatorname{AdS}_5 \times S^5$ string S -matrix [27, 28] and

$$\begin{aligned}
S_0(p_1, p_2) &= \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \sigma(p_1, p_2), \\
\tilde{S}_0(p_1, p_2) &= \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \sigma(p_1, p_2).
\end{aligned} \tag{3.2}$$

In (3.2) $\sigma(p_1, p_2)$ is the BES dressing factor [29, 30]. The S -matrices in (3.1) satisfy the unitarity condition ($S_{AA}^{12} S_{AA}^{21} = \mathbb{I}$) and the Yang-Baxter equations. In particular, the scalar factors S_0, \tilde{S}_0 fulfil separately the unitary constraint and obey the following crossing relations:

$$S_0(p_1, p_2) \tilde{S}_0(\bar{p}_1, p_2) = S_0(p_1, p_2) \tilde{S}_0(p_1, \bar{p}_2) = f(p_1, p_2), \tag{3.3}$$

with

$$f(p_1, p_2) = \frac{\left(\frac{1}{x_1^-} - x_2^-\right)(x_1^- - x_2^+)}{\left(\frac{1}{x_1^+} - x_2^-\right)(x_1^+ - x_2^+)}, \tag{3.4}$$

and $x^\pm(\bar{p}) = 1/x^\pm(p)$. Using the elliptic parametrisation equation (3.3) becomes

$$S_0(z_1, z_2) \tilde{S}_0(z_1 + \omega_2, z_2) = S_0(z_1, z_2) \tilde{S}_0(z_1, z_2 - \omega_2) = f(z_1, z_2). \tag{3.5}$$

Since $S(p_1, p_2)$ coincides with the $SU(2|2)$ S -matrix of $\text{AdS}_5/\text{CFT}_4$, one can repeat the analysis of [20] concerning the underlying supersymmetry algebra, the parametrisation of this S -matrix and its mirror version. The only difference occurs when we consider the scalar factor: as we can see in (3.3), the crossing relation does not relate a S -matrix with itself, but rather S_{AA} (S_{BB}) with S_{AB} (S_{BA}) and vice versa. However, the S -matrices (3.1) continue to satisfy the invariance under simultaneous shift of the arguments z_1 and z_2 by one half of the imaginary period:

$$\begin{aligned}
S_{AA}(z_1 + \omega_2, z_2 + \omega_2) &= S_{BB}(z_1, z_2) = S_{AA}(z_1, z_2), \\
S_{AB}(z_1 + \omega_2, z_2 + \omega_2) &= S_{BA}(z_1, z_2) = S_{AB}(z_1, z_2).
\end{aligned}$$

Then we assume that the S -matrices (3.1) share the good properties of the $SU(2|2)$ -related scattering matrix $S(p_1, p_2)$ and admit an analytic continuation in the z -variables, such that the mirror S -matrices are given by

$$\begin{aligned}
\tilde{S}_{AA}(z_1, z_2) &= \tilde{S}_{BB}(z_1, z_2) = S_{AA}\left(z_1 + \frac{\omega_2}{2}, z_2 + \frac{\omega_2}{2}\right) = S_{BB}\left(z_1 + \frac{\omega_2}{2}, z_2 + \frac{\omega_2}{2}\right), \\
\tilde{S}_{AB}(z_1, z_2) &= \tilde{S}_{BA}(z_1, z_2) = S_{AB}\left(z_1 + \frac{\omega_2}{2}, z_2 + \frac{\omega_2}{2}\right) = S_{BA}\left(z_1 + \frac{\omega_2}{2}, z_2 + \frac{\omega_2}{2}\right).
\end{aligned}$$

They result still to satisfy the generalised unitarity condition

$$\left[\tilde{S}_{AA}(z_1, z_2)\right]^\dagger = \tilde{S}_{AA}(z_2^*, z_1^*),$$

and the dressing factor in the mirror theory has been derived in [31].

4 The Bethe Ansatz equations

The asymptotic Bethe Ansatz equations for the $\mathcal{N} = 6$ superconformal Chern-Simons theory are [6]

$$\begin{aligned}
e^{ip_k^A J} &= \prod_{\substack{l=1 \\ l \neq k}}^{K_A^I} \left(\frac{u_k^A - u_l^A + \frac{2i}{h}}{u_k^A - u_l^A - \frac{2i}{h}} \right) \left(\frac{x_k^{A-} - x_l^{A+}}{x_k^{A+} - x_l^{A-}} \right)^{\frac{1-\eta}{2}} \left(\sqrt{\frac{x_l^{A+} x_k^{A-}}{x_l^{A-} x_k^{A+}}} \right)^{\frac{1+\eta}{2}} \sigma(p_k^A, p_l^A) \\
&\times \prod_{l=1}^{K_B^I} \left(\frac{x_k^{A-} - x_l^{B+}}{x_k^{A+} - x_l^{B-}} \right)^{\frac{1-\eta}{2}} \left(\sqrt{\frac{x_l^{B+} x_k^{B-}}{x_l^{B-} x_k^{B+}}} \right)^{\frac{1+\eta}{2}} \sigma(p_k^A, p_l^B) \prod_{j=1}^{K^{II}} \left(\frac{x_k^{A-} - y_j}{x_k^{A+} - y_j} \right) \sqrt{\frac{x_k^{A+}}{x_k^{A-}}}^\eta, \\
&(k = 1, \dots, K_A^I)
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
e^{ip_k^B J} &= \prod_{\substack{l=1 \\ l \neq k}}^{K_B^I} \left(\frac{u_k^B - u_l^B + \frac{2i}{h}}{u_k^B - u_l^B - \frac{2i}{h}} \right) \left(\frac{x_k^{B-} - x_l^{B+}}{x_k^{B+} - x_l^{B-}} \right)^{\frac{1-\eta}{2}} \left(\sqrt{\frac{x_l^{B+} x_k^{B-}}{x_l^{B-} x_k^{B+}}} \right)^{\frac{1+\eta}{2}} \sigma(p_k^B, p_l^B) \\
&\times \prod_{l=1}^{K_A^I} \left(\frac{x_k^{B-} - x_l^{A+}}{x_k^{B+} - x_l^{A-}} \right)^{\frac{1-\eta}{2}} \left(\sqrt{\frac{x_l^{A+} x_k^{A-}}{x_l^{A-} x_k^{A+}}} \right)^{\frac{1+\eta}{2}} \sigma(p_k^B, p_l^A) \prod_{j=1}^{K^{II}} \left(\frac{x_k^{B-} - y_j}{x_k^{B+} - y_j} \sqrt{\frac{x_k^{B+}}{x_k^{B-}}} \right)^\eta, \\
&(k = 1, \dots, K_B^I)
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
1 &= \prod_{l=1}^{K_A^I} \left(\frac{y_k - x_l^{A+}}{y_k - x_l^{A-}} \right) \sqrt{\frac{x_l^{A-}}{x_l^{A+}}} \prod_{\substack{l=1 \\ l \neq k}}^{K_B^I} \left(\frac{y_k - x_l^{B+}}{y_k - x_l^{B-}} \right) \sqrt{\frac{x_l^{B-}}{x_l^{B+}}} \prod_{l=1}^{K^{III}} \left(\frac{v_k - w_l + \frac{i}{h}}{v_k - w_l - \frac{i}{h}} \right), \\
&(k = 1, \dots, K^{II}),
\end{aligned} \tag{4.3}$$

$$1 = \prod_{l=1}^{K^{II}} \left(\frac{w_k - v_l - \frac{i}{h}}{w_k - v_l + \frac{i}{h}} \right) \prod_{\substack{l=1 \\ l \neq k}}^{K^{III}} \left(\frac{w_k - w_l + \frac{2i}{h}}{w_k - w_l - \frac{2i}{h}} \right), \quad (k = 1, \dots, K^{III}), \tag{4.4}$$

where

$$x_k^{\alpha \pm} = \frac{u_k^\alpha \pm \frac{i}{h}}{2} \left(1 + \sqrt{1 - \frac{4}{(u_k^\alpha \pm \frac{i}{h})^2}} \right), \tag{4.5}$$

with $\alpha = A, B$, $h = h(\lambda)$ and $\eta = \pm 1$ is the grading related to two different Dynkin diagrams. For $\eta = 1$ the momentum-carrying Bethe roots are bosonic, they can form single species bound states and the two massive nodes are linked to each other only through a single BES dressing factor. In the $\eta = -1$ case, instead, the momentum-carrying roots are fermionic and cannot form bound states in the physical theory.

We see that the form of the asymptotic Bethe Ansatz equations crucially depends on the grading η or, equivalently, on the choice of the reference state. In [20] the $sl(2)$ grading $\eta = -1$

was chosen for the mirror BA equations. Similarly, we choose here the $sl(2|1)$ grading for the mirror BAEs:

$$e^{i\tilde{p}_k^A R} = \prod_{l=1}^{K_A^I} S_0(\tilde{p}_k^A, \tilde{p}_l^A) \prod_{l=1}^{K_B^I} \tilde{S}_0(\tilde{p}_k^A, \tilde{p}_l^B) \prod_{l=1}^{K^{II}} \left(\frac{x_k^{A+} - y_l}{x_k^{-A} - y_l} \right) \sqrt{\frac{x_k^{A-}}{x_k^{A+}}}, \quad (4.6)$$

$$e^{i\tilde{p}_k^B R} = \prod_{l=1}^{K_B^I} S_0(\tilde{p}_k^B, \tilde{p}_l^B) \prod_{l=1}^{K_A^I} \tilde{S}_0(\tilde{p}_k^B, \tilde{p}_l^A) \prod_{l=1}^{K^{II}} \left(\frac{x_k^{B+} - y_l}{x_k^{B-} - y_l} \right) \sqrt{\frac{x_k^{B-}}{x_k^{B+}}}, \quad (4.7)$$

$$-1 = \prod_{\alpha=A,B} \prod_{l=1}^{K_\alpha^I} \left(\frac{y_k - x_l^{\alpha+}}{y_k - x_l^{\alpha-}} \right) \sqrt{\frac{x_l^{\alpha-}}{x_l^{\alpha+}}} \prod_{l=1}^{K^{III}} \left(\frac{v_k - w_l + \frac{i}{h}}{v_k - w_l - \frac{i}{h}} \right), \quad (4.8)$$

$$1 = \prod_{l=1}^{K^{II}} \left(\frac{w_k - v_l - \frac{i}{h}}{w_k - v_l + \frac{i}{h}} \right) \prod_{\substack{l=1 \\ l \neq k}}^{K^{III}} \left(\frac{w_k - w_l + \frac{2i}{h}}{w_k - w_l - \frac{2i}{h}} \right), \quad (4.9)$$

where

$$S_0(\tilde{p}_k^\alpha, \tilde{p}_l^\alpha) = \left(\frac{u_k^\alpha - u_l^\alpha + \frac{2i}{h}}{u_k^\alpha - u_l^\alpha - \frac{2i}{h}} \right) \left(\frac{x_k^{\alpha-} - x_l^{\alpha+}}{x_k^{\alpha+} - x_l^{\alpha-}} \right) \sigma(\tilde{p}_k^\alpha, \tilde{p}_l^\alpha), \quad (4.10)$$

$$\tilde{S}_0(\tilde{p}_k^\alpha, \tilde{p}_l^\beta) = \left(\frac{x_k^{\alpha-} - x_l^{\beta+}}{x_k^{\alpha+} - x_l^{\beta-}} \right) \sigma(\tilde{p}_k^\alpha, \tilde{p}_l^\beta). \quad (4.11)$$

Notice that for the y -particles, $y_k = ie^{-iq_k}$ and exchanging $q_k \leftrightarrow \pi - q_k \bmod(2\pi)$ corresponds to $y_k \leftrightarrow 1/y_k$. Considering the latter property, and in order to write the final equations in a more transparent form, it is convenient to define the multi-valued function $y(u)$ such that

$$y(u) = \begin{cases} x(u) & \text{for } \Im m(y) < 0, \\ 1/x(u) & \text{for } \Im m(y) > 0, \end{cases} \quad (4.12)$$

where $x(u)$ is defined by

$$x(u) = \left(\frac{u}{2} - i\sqrt{1 - \frac{u^2}{4}} \right), \quad (4.13)$$

and the property $x(u) = 1/x^*(u^*)$. Setting $u(q) = 2\sin(q)$ and

$$u_k^{y|+} = u(q_k), \quad (4.14)$$

with $q_k \in (-\pi/2, \pi/2]$, or equivalently $\Im m(y_k) > 0$, we have

$$y_k = y(u_k^{y|+}) \equiv 1/x(u(q_k)). \quad (4.15)$$

Similarly for $q_k \in (-\pi, -\pi/2] \cup (\pi/2, \pi]$, or equivalently $\Im m(y_k) < 0$:

$$y_k = y(u_k^{y|-}) = y((u(q_k) + 2)e^{i2\pi} - 2) \equiv x(u(q_k)). \quad (4.16)$$

5 The string hypothesis

Considering the pole structure of equations (4.6-4.9) and the mathematical resemblance between equations (4.8) and (4.9) with those of an inhomogeneous Hubbard model, we can formulate a string hypothesis for the solutions, in strict analogy with the Takahashi's one [32]. We shall assume that the thermodynamically relevant solutions of (4.6-4.9) in the limit of large $R, K_\alpha^I, K^{II}, K^{III}$ rearrange themselves into complexes – the so-called strings – with real centres and all the other complex roots symmetrically distributed around these centres along the imaginary direction.

In particular, the strings of the ‘massive’ Bethe roots are those of an $sl(2|1)$ spin chain (see [33] for a discussion on the string hypothesis and the TBA for this model), with mixed, alternating configurations of roots of kind A and B . In fact, they arise from considering the following parts of the equations (4.6) and (4.7), where we set all the auxiliary roots to zero:

$$e^{i\tilde{p}_k^A R} = \prod_{l=1}^{K_A^I} \frac{1 - \frac{1}{x_k^{A+} x_l^{A-}}}{1 - \frac{1}{x_k^{A-} x_l^{A+}}} \sigma(\tilde{p}_k^A, \tilde{p}_l^A) \prod_{l=1}^{K_B^I} \frac{x_k^{A-} - x_l^{B+}}{x_k^{A+} - x_l^{B-}} \sigma(\tilde{p}_k^A, \tilde{p}_l^B), \quad (5.1)$$

$$e^{i\tilde{p}_k^B R} = \prod_{l=1}^{K_B^I} \frac{1 - \frac{1}{x_k^{B+} x_l^{B-}}}{1 - \frac{1}{x_k^{B-} x_l^{B+}}} \sigma(\tilde{p}_k^B, \tilde{p}_l^B) \prod_{l=1}^{K_A^I} \frac{x_k^{B-} - x_l^{A+}}{x_k^{B+} - x_l^{A-}} \sigma(\tilde{p}_k^B, \tilde{p}_l^A). \quad (5.2)$$

When $R \rightarrow \infty$, the following two-particle bound state equations hold (let us suppress the indexes $k = l$ for simplicity)

$$x^{A+} - x^{B-} = 0 \quad \text{or} \quad x_1^{B+} - x_1^{A-} = 0. \quad (5.3)$$

The first equation is solved by $\tilde{p}^A = p/2 - iq, \tilde{p}^B = p/2 + iq$, with $\Re(q) > 0$, or $u^B - u^A = 2i/h$, the second one by $u^A - u^B = 2i/h$. We verified that the possible bound states equations arising from the other denominators, of the type $1 - \frac{1}{x_k^{\alpha-} x_l^{\alpha+}} = 0$, do not give ‘good solutions’ and assume that also the poles of the dressing factor do not give physical bound states.

Thus, thanks to the great similarity of the remaining part of the equations (4.6) with the BA equations of the $\text{AdS}_5 \times S^5$ mirror model [20, 25] and considering that

$$\tilde{p}_k^Q = \tilde{p}^Q(u_k) \equiv \tilde{p}(u_k) = \frac{i\hbar}{2} \left(\sqrt{4 - \left(u_k + i\frac{Q}{\hbar}\right)^2} - \sqrt{4 - \left(u_k - i\frac{Q}{\hbar}\right)^2} \right), \quad (5.4)$$

we can propose the following string classification:

- Wide strings

I) $N_{Q|WI}$ WI -particles (bound states) with real momenta $\tilde{p}_k^{Q|WI} = \tilde{p}_k^{2Q-1}$ and real rapidities $u_k^{Q|WI}$:

$$u_{k,j}^{AQ} = u_k^{Q|WI} + \frac{i}{h} (2Q + 2 - 4j), \quad j = 1, \dots, Q; \quad (5.5)$$

$$u_{k,l}^{BQ} = u_k^{Q|WI} + \frac{i}{h} (2Q - 4l), \quad l = 1, \dots, Q - 1. \quad (5.6)$$

II) $N_{Q|WII}$ WII -particles (bound states) with real momenta $\tilde{p}_k^{Q|WII} = \tilde{p}_k^{2Q-1}$ and real rapidities $u_k^{Q|WII}$:

$$u_{k,j}^{AQ} = u_k^{Q|WII} + \frac{i}{h} (2Q - 4j) , \quad j = 1, \dots, Q - 1 ; \quad (5.7)$$

$$u_{k,l}^{BQ} = u_k^{Q|WII} + \frac{i}{h} (2Q + 2 - 4l) , \quad l = 1, \dots, Q . \quad (5.8)$$

- Strange strings

I) $N_{Q|sI}$ sI -particles (bound states) with real momenta $\tilde{p}_k^{Q|sI} = \tilde{p}_k^{2Q}$ and real rapidities $u_k^{Q|sI}$:

$$u_{k,j}^{AQ} = u_k^{Q|sI} - \frac{i}{h} (2Q + 1 - 4j) , \quad j = 1, \dots, Q ; \quad (5.9)$$

$$u_{k,l}^{BQ} = u_k^{Q|sI} + \frac{i}{h} (2Q + 1 - 4l) , \quad l = 1, \dots, Q . \quad (5.10)$$

II) $N_{Q|sII}$ sII -particles (bound states) with real momenta $\tilde{p}_k^{Q|sII}$ and real rapidities $u_k^{Q|sII}$:

$$u_{k,j}^{AQ} = u_k^{Q|sII} + \frac{i}{h} (2Q + 1 - 4j) , \quad j = 1, \dots, Q ; \quad (5.11)$$

$$u_{k,l}^{BQ} = u_k^{Q|sII} - \frac{i}{h} (2Q + 1 - 4l) , \quad l = 1, \dots, Q . \quad (5.12)$$

- N_y y -particles with real momenta $q_k \in (-\pi, \pi]$.

- $N_{M|v}$ vw -strings with real centers v_k^M , $2M$ roots of type v and M of type w :

$$v_k^{M,j} = v_k^M \pm \frac{i}{h} (M + 2 - 2j) , \quad j = 1, \dots, M ; \quad (5.13)$$

$$w_k^{M,j} = v_k^M + \frac{i}{h} (M + 1 - 2j) , \quad j = 1, \dots, M . \quad (5.14)$$

- $N_{N|w}$ w -strings with real centres w_k^N and N roots of type w :

$$w_k^{N,j} = w_k^N + \frac{i}{h} (N + 1 - 2j) , \quad j = 1, \dots, N . \quad (5.15)$$

Notice that the strings WI have Bethe roots of type A as real centres, the WII ones have Bethe roots of type B as real centres; the strange strings have not real centres belonging to the Bethe roots and are not invariant under complex conjugation. We will return on this peculiar property in the next section, when we will discuss the root density equations. Replacing the variables $u_k^{A,B}$, v_k and w_k in (4.6) with $u_{k,j}^Q$, $v_k^{M,j}$ and $w_k^{M,j}$, performing the products on the internal string index j , and relabelling the Q -related quantities as

$$\mathcal{Q}|\alpha = \begin{cases} Q|W\alpha & \text{for } \mathcal{Q} = 2Q - 1 ; \\ Q|s\alpha & \text{for } \mathcal{Q} = 2Q , \end{cases} \quad (5.16)$$

we arrive to the following equations for the real centres of the strings (5.5-5.15):

$$\begin{aligned}
1 &= e^{i\tilde{p}_k^{\mathcal{Q}|\alpha} R} \prod_{\beta, \mathcal{Q}'} \prod_{l=1}^{N_{\mathcal{Q}'|\beta}} S_{(\mathcal{Q}|\alpha), (\mathcal{Q}'|\beta)}(u_k^{\mathcal{Q}|\alpha}, u_l^{\mathcal{Q}'|\beta}) \prod_M \prod_{l=1}^{N_{M|vw}} S_{\mathcal{Q}, (v|M)}(u_k^{\mathcal{Q}|\alpha}, v_l^M) \\
&\times \prod_{\delta=\pm} \prod_{l=1}^{N_{y|\delta}} S_{\mathcal{Q}, y}(u_k^{\mathcal{Q}|\alpha}, u_l^{y|\delta}), \tag{5.17}
\end{aligned}$$

$$-1 = \prod_{\beta, \mathcal{Q}} \prod_{l=1}^{N_{\mathcal{Q}|\beta}} S_{y, \mathcal{Q}}(u_k^{y|\pm}, u_l^{\mathcal{Q}|\beta}) \prod_M \prod_{l=1}^{N_{M|w}} S_M(u_k^{y|\pm} - w_l^M) \prod_{l=1}^{N_{M|vw}} S_M(u_k^{y|\pm} - v_l^M), \tag{5.18}$$

$$(-1)^K = \prod_{\beta, \mathcal{Q}} \prod_{l=1}^{N_{\mathcal{Q}|\beta}} S_{(v|K), \mathcal{Q}}(v_k^K, u_l^{\mathcal{Q}|\beta}) \prod_M \prod_{l=1}^{N_{M|vw}} S_{KM}(v_k^K - v_l^M) \prod_{\delta=\pm} \prod_{l=1}^{N_{y|\delta}} S_K(v_k^K - u_l^{y|\delta}), \tag{5.19}$$

$$(-1)^K = \prod_M \prod_{l=1}^{N_{w|M}} S_{KM}(w_k^K - w_l^M) \prod_{\delta=\pm} \prod_{l=1}^{N_{y|\delta}} \left(S_K(w_k^K - u_l^{y|\delta}) \right)^{-1}, \tag{5.20}$$

with $\mathcal{Q} = 1, 2, \dots$ and $\alpha = I, II$. The scalar $S_{AB}(u, z)$ factors in (5.17–5.20) are listed in Appendix A.

6 The thermodynamic Bethe Ansatz method

The mirror Bethe Ansatz equations (5.17-5.20) can be used to derive, using a procedure [13, 32, 34, 14] already successfully adapted to the study of $\text{AdS}_5/\text{CFT}_4$ in [10, 11, 12], a set of thermodynamic Bethe Ansatz equations describing the ground-state of the (direct) $\text{AdS}_4/\text{CFT}_3$ theory. Taking the logarithm of (5.17-5.20), and introducing the collective index A for the different density labels we can perform the thermodynamic limit $N_A, R \rightarrow \infty$ with N_A/R finite. The density of states ρ_A is

$$\rho_A(u) = \rho_A^r(u) + \rho_A^h(u) = \lim_{R \rightarrow \infty} \frac{I_{k+1}^A - I_k^A}{R(u_{k+1}^A - u_k^A)}, \tag{6.1}$$

where ρ_A^r and ρ_A^h are respectively the density of roots and holes and the I 's are the Bethe quantum numbers. $I_k \in \mathbb{Z}$ for $A \in \{(\mathcal{Q}|\alpha), (y|\pm)\}$ with $\alpha = I, II$ and $\mathcal{Q} = 1, 2, \dots$ while $I_k \in \mathbb{Z} + 1/2$ for $A \in \{(v|K), (w|K)\}$ with $K = 1, 2, \dots$.

The Bethe Ansatz equations (5.17–5.20) lead to a set of constraints for the densities (6.1):

$$\begin{aligned} \rho_{Q|\alpha}(u) &= \frac{1}{2\pi} \frac{d\tilde{p}^Q(u)}{du} + \sum_{\beta} \sum_{Q'=1}^{\infty} \phi_{(Q|\alpha), (Q'|\beta)} * \rho_{Q'|\beta}^r(u) + \sum_{M=1}^{\infty} \phi_{Q, (v|M)} * \rho_{v|M}^r(u) \\ &+ \int_{-2}^2 dz \left[\phi_{Q, (y| -)}(u, z) \rho_{y| -}^r(z) + \phi_{Q, (y| +)}(u, z) \rho_{y| +}^r(z) \right] , \end{aligned} \quad (6.2)$$

$$\rho_{y| -}(u) = - \sum_{Q=1}^{\infty} \phi_{(y| -), Q} * (\rho_{Q|I}^r(u) + \rho_{Q|II}^r(u)) - \sum_{M=1}^{\infty} \phi_M * (\rho_{w|M}^r(u) + \rho_{v|M}^r(u)) , \quad (6.3)$$

$$\begin{aligned} \rho_{v|K}(u) &= - \sum_{Q=1}^{\infty} \phi_{(v|K), Q} * (\rho_{Q|I}^r(u) + \rho_{Q|II}^r(u)) - \sum_{M=1}^{\infty} \phi_{K,M} * \rho_{v|M}^r(u) \\ &- \int_{-2}^2 dz \phi_K(u - z) (\rho_{y| -}^r(z) + \rho_{y| +}^r(z)) , \end{aligned} \quad (6.4)$$

$$\rho_{w|K}(u) = - \sum_{M=1}^{\infty} \phi_{K,M} * \rho_{w|M}^r(u) + \int_{-2}^2 dz \phi_K(u - z) (\rho_{y| -}^r(z) + \rho_{y| +}^r(z)) , \quad (6.5)$$

where

$$\rho_{y| +}(u) = \rho_{y| -}((u + 2)e^{i2\pi} - 2) , \quad \rho_{y| -}^r(u) = \rho_{y| +}^r((u + 2)e^{i2\pi} - 2) , \quad (6.6)$$

and the symbol ‘*’ denotes the convolution

$$\phi * \rho(u) = \int_{\mathbb{R}} dz \phi(u, z) \rho(z) . \quad (6.7)$$

The kernels are

$$\phi_{AB}(z, u) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \ln S_{AB}(z, u) . \quad (6.8)$$

Finally, as anticipated in the previous section, the strange strings are not invariant under complex conjugation and to define real densities $\rho_{2Q|\alpha}(u)$, we have to impose the equality between the real centres of the strange strings of type I and type II. This corresponds to

$$\rho_{2Q|I}(u) = \rho_{2Q|II}(u) . \quad (6.9)$$

6.1 The TBA equations

In terms of hole and root densities, the entropy is

$$S = \sum_A \int du \left((\rho_A^r(u) + \rho_A^h(u)) \ln(\rho_A^r(u) + \rho_A^h(u)) - \rho_A^r(u) \ln \rho_A^r(u) \right) , \quad (6.10)$$

and the free energy per unit length:

$$f(T) = \tilde{H} - TS . \quad (6.11)$$

In (6.11) \tilde{H} is the (mirror) energy per unit length:

$$\tilde{H} = \sum_{Q=1}^{\infty} \int_{\mathbb{R}} du E_Q(u) (\rho_{Q|I}^r(u) + \rho_{Q|II}^r(u)) , \quad (6.12)$$

with

$$E_Q(u) = \ln \frac{x(u - \frac{i}{h}Q)}{x(u + \frac{i}{h}Q)} . \quad (6.13)$$

The temperature T of the mirror theory corresponds to the inverse of the trace operator length L in $\mathcal{N} = 6$ superconformal Chern-Simons theory. The extremum condition $\delta f = 0$ under the constraints (6.2-6.5) leads to the set of TBA equations for the pseudoenergies $\varepsilon_A(u)$:

$$\varepsilon_A(u) = \ln \frac{\rho_A^h(u)}{\rho_A^r(u)} , \quad \frac{1}{e^{\varepsilon_A(u)} + 1} = \frac{\rho_A^r(u)}{\rho_A(u)} , \quad L_A(u) = \ln \left(1 + e^{-\varepsilon_A(u)} \right) . \quad (6.14)$$

The TBA equations are

$$\begin{aligned} \varepsilon_{Q|\alpha}(u) &= L E_Q(u) - \sum_{\beta} \sum_{Q'=1}^{\infty} L_{Q'|\beta} * \phi_{(Q'|\beta), (Q|\alpha)}(u) + \sum_{M=1}^{\infty} L_{v|M} * \phi_{(v|M), Q}(u) \\ &+ \int_{-2}^2 dz [L_{y|-}(z) \phi_{(y|-), Q}(z, u) - L_{y|+}(z) \phi_{(y|+), Q}(z, u)] , \end{aligned} \quad (6.15)$$

$$\varepsilon_{y|-}(u) = - \sum_{Q=1}^{\infty} (L_{Q|I} + L_{Q|II}) * \phi_{Q, (y|-)}(u) + \sum_{M=1}^{\infty} (L_{v|N} - L_{w|M}) * \phi_M(u) , \quad (6.16)$$

$$\begin{aligned} \varepsilon_{v|K}(u) &= - \sum_{Q=1}^{\infty} (L_{Q|I} + L_{Q|II}) * \phi_{Q, (v|K)}(u) + \sum_{M=1}^{\infty} L_{v|M} * \phi_{M, K}(u) \\ &+ \int_{-2}^2 dz (L_{y|-}(z) - L_{y|+}(z)) \phi_K(z - u) , \end{aligned} \quad (6.17)$$

$$\varepsilon_{w|K}(u) = \sum_{M=1}^{\infty} L_{w|M} * \phi_{M, K}(u) + \int_{-2}^2 dz (L_{y|-}(z) - L_{y|+}(z)) \phi_K(z - u) , \quad (6.18)$$

where

$$\varepsilon_{y|+}(u) = \varepsilon_{y|-}((u + 2)e^{i2\pi} - 2) , \quad (6.19)$$

and

$$L * \phi(u) = \int_{\mathbb{R}} dz L(z) \phi(z, u) . \quad (6.20)$$

Finally, the *minimal* free energy is given by the following non-linear functional of the pseudoenergies $\varepsilon_{Q|\alpha}(u)$

$$f(T) = -T \sum_{Q=1}^{\infty} \int_{\mathbb{R}} \frac{du}{2\pi} \frac{d\tilde{p}^Q}{du} (L_{Q|I}(u) + L_{Q|II}(u)) , \quad (6.21)$$

where $f(T)$ is related to the ground state energy for the AdS/CFT theory on a circumference with length $L = 1/T$ by the relation

$$E_0(L) = Lf(1/L) . \quad (6.22)$$

As we have kept the total densities finite, it is natural to introduce chemical potentials μ_A . For relativistic theories this has been discussed in [35]. The TBA equations (6.15–6.18) do not change their form, but for this simple replacement

$$L_A = \ln(1 + e^{-\epsilon_A}) \rightarrow L_{A,\lambda} = \ln(1 + \lambda_A e^{-\epsilon_A}) , \quad (6.23)$$

involving the fugacities $\lambda_A = e^{\mu_A/T}$.

In our case we expect zero energy as soon as the fugacities reach these values

$$\lambda_{\mathcal{Q}|\alpha} = (-1)^{\mathcal{Q}} , \quad \lambda_{v|K} = \lambda_{w|K} = 1 , \quad \lambda_{y|\pm} = -1 , \quad (\alpha = I, II, K = 1, 2, \dots) . \quad (6.24)$$

Physically, this modification corresponds to the computation of the Witten index and the proposal (6.24) reflects the bosonic/fermionic character of the various excitations. A vanishing ground state energy can be given by a singularity in the solution of the massive TBA equation (6.15). Of course, the latter needs to be regularised by introducing the chemical potentials

$$\lambda_{2\mathcal{Q}-1|I} = -e^{ih} , \quad \lambda_{2\mathcal{Q}-1|II} = -e^{-ih} , \quad \lambda_{2\mathcal{Q}|I} = \lambda_{2\mathcal{Q}|II} = 1 , \quad \lambda_{v|K} = \lambda_{w|K} = 1 , \quad \lambda_{y|\pm} = -1 , \quad (6.25)$$

such that the TBA equations are regular for $h \neq 0$ and the ground state energy tends to zero as $h \rightarrow 0$. It would be important to check directly the vanishing of $E_0(L)$ using numerical or analytic methods.

7 The Y-system

Using the methods adopted in [36, 10, 12] and the kernel identities listed in Appendix A, the following Y-system valid in the strip $|\Re(u)| < 2$ can be derived. For the $WI-$, $WII-$, $sI-$ and $sII-$ related pseudoenergies we have

$$Y_{1|I}(u + \frac{i}{h})Y_{1|II}(u - \frac{i}{h}) = (1 + Y_{2|I}(u)) \left(1 + \frac{1}{Y_{y|-}(u)}\right)^{-1} , \quad (7.26)$$

$$Y_{1|II}(u + \frac{i}{h})Y_{1|I}(u - \frac{i}{h}) = (1 + Y_{2|II}(u)) \left(1 + \frac{1}{Y_{y|-}(u)}\right)^{-1} , \quad (7.27)$$

$$Y_{\mathcal{Q}|I}(u + \frac{i}{h})Y_{\mathcal{Q}|II}(u - \frac{i}{h}) = (1 + Y_{\mathcal{Q}+1|I}(u))(1 + Y_{\mathcal{Q}-1|II}(u)) \left(1 + \frac{1}{Y_{v|\mathcal{Q}-1}(u)}\right)^{-1} , \quad (7.28)$$

$$Y_{\mathcal{Q}|II}(u + \frac{i}{h})Y_{\mathcal{Q}|I}(u - \frac{i}{h}) = (1 + Y_{\mathcal{Q}+1|II}(u))(1 + Y_{\mathcal{Q}-1|I}(u)) \left(1 + \frac{1}{Y_{v|\mathcal{Q}-1}(u)}\right)^{-1} , \quad (7.29)$$

with $Y_A = e^{\epsilon_A}$ and $\mathcal{Q} = 2, 3, \dots$. The reader should notice that equations (7.26-7.29) have a slightly different structure compared to the standard Y-systems as, for example, those proposed for the same model in [26]. The right-hand-sides of (7.26-7.29) involve mixed pairs of type *I* and type *II* functions. A similar non-standard structure has been previously observed in the context of a much-simpler D_n -related family of Y-systems [37]. In the current case we suspect this property should be directly related to the multi-valued character of the Y-functions. The equations for the remaining TBA nodes are:

$$Y_{w|K}(u + \frac{i}{h})Y_{w|K}(u - \frac{i}{h}) = \prod_{M=1}^{\infty} (1 + Y_{w|M}(u))^{I_{KM}} \left(\frac{1 + \frac{1}{Y_{y|-}(u)}}{1 + \frac{1}{Y_{y|+}(u)}} \right)^{\delta_{K1}}, \quad (7.30)$$

$$Y_{y|-}(u + \frac{i}{h})Y_{y|-}(u - \frac{i}{h}) = \left(\frac{1 + Y_{v|1}(u)}{1 + Y_{w|1}(u)} \right) \prod_{\alpha=I,II} \left(1 + \frac{1}{Y_{1|\alpha}(u)} \right)^{-1}, \quad (7.31)$$

$$Y_{v|K}(u + \frac{i}{h})Y_{v|K}(u - \frac{i}{h}) = \frac{\prod_{M=1}^{\infty} (1 + Y_{v|M}(u))^{I_{KM}}}{\prod_{\alpha} \left(1 + \frac{1}{Y_{K+1|\alpha}(u)} \right)} \left(\frac{1 + Y_{y|-}(u)}{1 + Y_{y|+}(u)} \right)^{\delta_{K1}}, \quad (7.32)$$

with $K = 1, 2, \dots$. Although equations (7.26–7.32) were derived restricting u to the region $|\Re(u)| < 2$, they are obviously valid in a much wider region of the complex plane. However due to the presence of an infinite number of square-root branch points, the Y-functions are multi-valued and the analytic continuation of (7.26–7.32) outside the region $|\Re(u)| < 2$ requires special attention [12]. A more complete discussion of the analytic properties of the solutions of the TBA equations (6.15–6.18) is postponed to the near future.

8 Conclusions

The study of the anomalous dimensions of single trace composite operators in the planar $\mathcal{N} = 6$ superconformal Chern-Simons gauge theory in three dimensions is a very challenging objective. In this paper, following some analogies with the more studied $\mathcal{N} = 4$ super Yang-Mills example, we have proposed a set of all-loop Bethe Ansatz equations for the SCS *mirror* theory and formulated the corresponding string hypothesis. By means of these two ingredients it has been possible to derive a set of thermodynamic Bethe Ansatz equations for the ground-state energy and then an associated Y-system. Of course, the latter is less informative, and just because of its more generality it is conjectured to encompass excited state TBA, too.

The Y-system here clearly differs from that of [26] in the momentum carrying nodes. Actually, a Y-system like that proposed in [26] can be derived by applying the TBA procedure to the *direct* $\text{AdS}_4 \times \mathbb{CP}^3$ string theory. The latter is asymptotically described by the Bethe-Yang equations (4.4) in the grading $\eta = 1$, which imply $su(2)$ -like bound states for the A particles and, separately, for the B ones. Therefore, the string hypothesis would not be different from that in [25], but for the presence of two species of particles, A and B. Then, the Yang and Yang TBA procedure can be applied and the resulting TBA equations lead to a Y-system

like that in [26]. In this way, this Y -system is deeply related to ours above². Nevertheless, the two Y -systems share the same form as long as the massive nodes are identified, namely $Y_{Q|I}(u) = Y_{Q|II}(u)$ for the mirror case and $Y_{Q|A}(u) = Y_{Q|B}(u)$ in the direct case, where $Y_{Q|A}(u)$ and $Y_{Q|B}(u)$ are the Y s of the A - and B -bound states, respectively³. Pictorially, this procedure folds the two massive node chains into a unique one. Provided that the asymptotic expressions [26] for $Y_{Q|A}(u)$ and $Y_{Q|B}(u)$ of the irrepresentation **20** are part of the Y -system solution, they should be solution of the mirror Y -system as well.

In the framework of relativistic scattering models a number of tools have been developed over the years [15, 16, 17] to extend the equations from the ground state to the excited states. To verify the correctness and consistency of our proposals and to make the connection with the field-theory results the generalisation to excited states is almost compulsory. However, to achieve this objective, the analytic properties of the Y -functions should be understood at a deeper level. In particular the rôle of the dressing factor should be clarified. Some important progress in this direction have been recently made in [19], but we suspect that the AdS/CFT-related TBA equations contain many more interesting surprises.

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9 Appendix A

Here we report the scalar factors $S_{A,B}(u, z)$ involved in the Bethe Ansatz equations (5.17–5.20).

$$S_{y,Q}(u, z) = S_{Q,y}(z, u) = \left(\frac{x(z - \frac{i}{h}Q) - y(u)}{x(z + \frac{i}{h}Q) - y(u)} \right) \sqrt{\frac{x(z + \frac{i}{h}Q)}{x(z - \frac{i}{h}Q)}}. \quad (9.1)$$

The functions $S_{(y|\pm),Q}$ and $S_{Q,(y|\pm)}$ are related to (9.1) through equation (4.12), they are:

$$S_{(y|\mp),Q}(u, z) = S_{Q,(y|\mp)}(z, u) = \left(\frac{x(z - \frac{i}{h}Q) - (x(u))^{\pm 1}}{x(z + \frac{i}{h}Q) - (x(u))^{\pm 1}} \right) \sqrt{\frac{x(z + \frac{i}{h}Q)}{x(z - \frac{i}{h}Q)}}. \quad (9.2)$$

²Moreover, in the AdS₅/CFT₄ correspondence this difference in the functional form of the Y -system (between direct and mirror theory) does not subsist. We are not referring here to the other differences, like, for instance, the integration domains in the TBA equations.

³These are called respectively $Y_{a,0}^4(u)$ and $Y_{a,0}^{\bar{4}}(u)$ in [26].

Moreover, we may write down

$$S_{(v|M),\mathcal{Q}}(u, z) = S_{\mathcal{Q},(v|M)}(z, u) = \left(\frac{x(z - \frac{i}{h}\mathcal{Q}) - x(u + \frac{i}{h}M)}{x(z + \frac{i}{h}\mathcal{Q}) - x(u + \frac{i}{h}M)} \right) \left(\frac{x(z + \frac{i}{h}\mathcal{Q})}{x(z - \frac{i}{h}\mathcal{Q})} \right) \\ \times \left(\frac{x(z - \frac{i}{h}\mathcal{Q}) - x(u - \frac{i}{h}M)}{x(z + \frac{i}{h}\mathcal{Q}) - x(u - \frac{i}{h}M)} \right) \prod_{j=1}^{M-1} \left(\frac{z - u - \frac{i}{h}(\mathcal{Q} - M + 2j)}{z - u + \frac{i}{h}(\mathcal{Q} - M + 2j)} \right), \quad (9.3)$$

$$S_M(u) = \left(\frac{u - \frac{i}{h}M}{u + \frac{i}{h}M} \right), \quad (9.4)$$

$$S_{K,M}(u) = \left(\frac{u - \frac{i}{h}|K - M|}{u + \frac{i}{h}|K - M|} \right) \left(\frac{u - \frac{i}{h}(K + M)}{u + \frac{i}{h}(K + M)} \right) \prod_{k=1}^{\min(K,M)-1} \left(\frac{u - \frac{i}{h}(|K - M| + 2k)}{u + \frac{i}{h}(|K - M| + 2k)} \right)^2. \quad (9.5)$$

The elements $S_{(\mathcal{Q}|\alpha),(\mathcal{Q}'|\beta)}(u, z)$ are:

$$S_{(\mathcal{Q}|\alpha),(\mathcal{Q}'|\beta)}(u, z) = S_{(\mathcal{Q}|\alpha),(\mathcal{Q}'|\beta)}^0(u - z)(\Sigma^{\mathcal{Q},\mathcal{Q}'}(u, z))^{-1}, \quad (9.6)$$

where $\Sigma^{\mathcal{Q},\mathcal{Q}'}$ is the improved dressing factor for the mirror bound states defined and derived in [31]:

$$\Sigma^{\mathcal{Q},\mathcal{Q}'}(u, z) = \prod_{k=1}^{\mathcal{Q}} \prod_{l=1}^{\mathcal{Q}'} \left(\frac{1 - \frac{1}{x(u + \frac{i}{h}(\mathcal{Q} + 2 - 2k))x(z + \frac{i}{h}(\mathcal{Q}' - 2l))}}{1 - \frac{1}{x(u + \frac{i}{h}(\mathcal{Q} - 2k))x(z + \frac{i}{h}(\mathcal{Q}' + 2 - 2l))}} \right) \sigma^{\mathcal{Q},\mathcal{Q}'}(u, z). \quad (9.7)$$

Finally, for $\alpha \neq \beta$ ($\alpha' \neq \beta', \alpha'' \neq \beta'', \alpha''' \neq \beta'''$) we have:

$$S_{(2\mathcal{Q}-1|\alpha),(2\mathcal{Q}'-1|\alpha)}^0(u) = \left(\frac{u + \frac{2i}{h}|Q' - Q|}{u - \frac{2i}{h}|Q' - Q|} \right) \prod_{j=1}^{\min(Q,Q')-1} \left(\frac{u + \frac{2i}{h}(|Q' - Q| + 2j)}{u - \frac{2i}{h}(|Q' - Q| + 2j)} \right)^2, \quad (9.8)$$

$$S_{(2\mathcal{Q}|\alpha),(2\mathcal{Q}'|\alpha)}^0(u) = \left(\frac{u + \frac{2i}{h}(Q' + Q)}{u - \frac{2i}{h}(Q' + Q)} \right) \left(\frac{u + \frac{2i}{h}|Q' - Q|}{u - \frac{2i}{h}|Q' - Q|} \right) \\ \times \prod_{j=1}^{\min(Q,Q')-1} \left(\frac{u + \frac{2i}{h}(|Q' - Q| + 2j)}{u - \frac{2i}{h}(|Q' - Q| + 2j)} \right)^2, \quad (9.9)$$

$$S_{(2\mathcal{Q}-1|\alpha),(2\mathcal{Q}'-1|\beta)}^0(u) = \left(\frac{u + \frac{2i}{h}(Q' + Q - 1)}{u - \frac{2i}{h}(Q' + Q - 1)} \right) \\ \times \prod_{j=1}^{\min(Q,Q')-1} \left(\frac{u + \frac{2i}{h}(|Q' - Q| - 1 + 2j)}{u - \frac{2i}{h}(|Q' - Q| - 1 + 2j)} \right)^2, \quad (9.10)$$

$$S_{(2Q|\alpha),(2Q'|\beta)}^0(u) = \prod_{j=1}^{\min(Q,Q')} \left(\frac{u + \frac{2i}{h}(|Q' - Q| - 1 + 2j)}{u - \frac{2i}{h}(|Q' - Q| - 1 + 2j)} \right)^2, \quad (9.11)$$

$$\begin{aligned} S_{(2Q-1|\alpha),(2Q'|\alpha)}^0(u) &= \left(S_{(2Q-1|\alpha'),(2Q'|\beta')}^0(-u) \right)^{-1} = \left(\frac{u + \frac{2i}{h}(Q + Q' - \frac{1}{2})}{u - \frac{2i}{h}(Q' - Q + \frac{1}{2})} \right) \\ &\times \prod_{j=1}^{Q-1} \left(\frac{u + \frac{2i}{h}(Q' - Q - \frac{1}{2} + 2j)}{u - \frac{2i}{h}(Q' - Q + \frac{1}{2} + 2j)} \right)^2, \end{aligned} \quad (9.12)$$

$$\begin{aligned} S_{(2Q|\alpha),(2Q'-1|\alpha)}^0(u) &= \left(S_{(2Q|\alpha'),(2Q'-1|\beta')}^0(-u) \right)^{-1} = \left(S_{(2Q'-1|\alpha''),(2Q|\alpha'')}^0(-u) \right)^{-1} \\ &= S_{(2Q'-1|\alpha'''),(2Q|\beta''')}^0(u). \end{aligned} \quad (9.13)$$

Here we report the identities for the kernels useful for the derivation of the Y-system.

$$\begin{aligned} \phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}|\alpha)}\left(z, u + \frac{i}{h}\right) + \phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}|\beta)}\left(z, u - \frac{i}{h}\right) &= (\phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}+1|\alpha)} + \phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}-1|\beta)}) (z, u) \\ &- \delta(z-u)\delta_{\mathcal{Q}',\mathcal{Q}+1} , \end{aligned} \quad (9.14)$$

$$\begin{aligned} \phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}|\alpha)}\left(z, u - \frac{i}{h}\right) + \phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}|\beta)}\left(z, u + \frac{i}{h}\right) &= (\phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}-1|\alpha)} + \phi_{(\mathcal{Q}'|\alpha),(\mathcal{Q}+1|\beta)}) (z, u) \\ &- \delta(z-u)\delta_{\mathcal{Q}',\mathcal{Q}-1} , \end{aligned} \quad (9.15)$$

$$\begin{aligned} \phi_{(y|-,)\mathcal{Q}}\left(z, u + \frac{i}{h}\right) + \phi_{(y|-,)\mathcal{Q}}\left(z, u - \frac{i}{h}\right) &= \sum_{\mathcal{Q}'=1}^{\infty} I_{\mathcal{Q}\mathcal{Q}'}\phi_{(y|-,)\mathcal{Q}'}(z, u) \\ &+ \delta(z-u)\delta_{\mathcal{Q},1} , \end{aligned} \quad (9.16)$$

$$\phi_{(y|+)\mathcal{Q}}\left(z, u + \frac{i}{h}\right) + \phi_{(y|+)\mathcal{Q}}\left(z, u - \frac{i}{h}\right) = \sum_{\mathcal{Q}'=1}^{\infty} I_{\mathcal{Q}\mathcal{Q}'}\phi_{(y|+)\mathcal{Q}'}(z, u) , \quad (9.17)$$

$$\begin{aligned} \phi_{(v|M),\mathcal{Q}}\left(z, u + \frac{i}{h}\right) + \phi_{(v|M),\mathcal{Q}}\left(z, u - \frac{i}{h}\right) &= \sum_{\mathcal{Q}'=1}^{\infty} I_{\mathcal{Q}\mathcal{Q}'}\phi_{(v|M),\mathcal{Q}'}(z, u) \\ &- \delta(z-u)\delta_{\mathcal{Q}-1,M} , \end{aligned} \quad (9.18)$$

$$\phi_{\mathcal{Q},(y|-,)}\left(z, u + \frac{i}{h}\right) + \phi_{\mathcal{Q},(y|-,)}\left(z, u - \frac{i}{h}\right) = \phi_{\mathcal{Q},(v|1)}(z, u) + \delta(z-u)\delta_{\mathcal{Q},1} , \quad (9.19)$$

$$\begin{aligned} \phi_{\mathcal{Q},(v|M)}\left(z, u + \frac{i}{h}\right) + \phi_{\mathcal{Q},(v|M)}\left(z, u - \frac{i}{h}\right) &= \sum_{M'=1}^{\infty} I_{MM'}\phi_{\mathcal{Q},(v,M')}(z, u) \\ &+ \delta(z-u)\delta_{\mathcal{Q}-1,M} , \end{aligned} \quad (9.20)$$

$$\phi_{KM}\left(u + \frac{i}{h}\right) + \phi_{KM}\left(u - \frac{i}{h}\right) = \sum_{K'=1}^{\infty} I_{KK'}\phi_{K'M}(u) + I_{KM}\delta(u) , \quad (9.21)$$

$$\begin{aligned} \phi_M\left(u + \frac{i}{h}\right) + \phi_M\left(u - \frac{i}{h}\right) &= \phi_{M+1}(u) + \phi_{M-1}(u) + \delta_{M,1}\delta(u) \\ &= \phi_{M,1}(u) + \delta_{M,1}\delta(u) , \end{aligned} \quad (9.22)$$

where $I_{KK'} = \delta_{K+1,K'} + \delta_{K-1,K'}$.

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